UNIVERSITY of HOUSTON

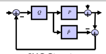
Design of Experiments for Multivariable System Identification Subject to Integral Controllability

UNIVERSITY of HOUSTON

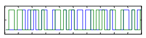
Mark L. Darby and Michael Nikolaou
Chemical and Biomolecular Engineering Dept.

Summary

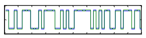
- **Problem Statement**: Identify model for MIMO controller
  - Satisfy Integral Controllability



- **Existing Approaches**
  - Well-conditioned systems: Use PRBS inputs



  - Ill-conditioned systems: Use correlated inputs based on uncorrelated rotated inputs



- **Open Issues**
  - Satisfy input/output constraints
  - Assume well or ill-conditioned system?
    - Use rotated or actual inputs?
  - How to distribute energy to inputs?

- **This Work**
  - Systematic approach to MIMO identification experiments subject to integral controllability
  - Main features of proposed approach
    - Satisfies input/output constraints
    - Does not require system characterization as well or ill-conditioned
    - Recovers existing MIMO input designs as special cases
    - Proposes new MIMO input designs
      - Generalized rotated inputs
    - Estimates upper bound for experiment duration

Integral Controllability

- MIMO model identified from an identification experiment using a least-squares method
- Model-based control needs integral controllability

Garcia & Morari (1985)

$\mathrm{Re}\left[\lambda\left(\mathbf{G}\hat{\mathbf{G}}^{-1}\right)\right]>0 \Leftrightarrow$ There exists a robustly stabilizing controller with integral action

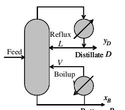
$\mathbf{G}$ = steady-state gain matrix, true plant
$\hat{\mathbf{G}}$ = steady-state gain matrix, estimate

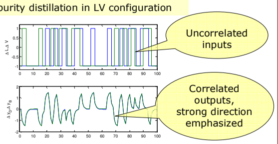
- A challenge for ill-conditioned systems
- Sub-matrices of MPC model must satisfy

Ill-Conditioned Systems

- Characterized by large condition number and significant gain variation with input directions
- Use of PRBS often leads to models that perform poorly in closed loop

Example: High purity distillation in LV configuration





$$\begin{bmatrix} y_1\\ y_2\end{bmatrix}=\mathbf{G}\begin{bmatrix} m_1\\ m_2\end{bmatrix}=\mathbf{U\Sigma V}^T\mathbf{m}=(\sigma_1\mathbf{u}_1\mathbf{v}_1^T+\sigma_2\mathbf{u}_2\mathbf{v}_2^T)\mathbf{m}\overset{\sigma_1>>\sigma_2}{\approx}\sigma_1\mathbf{u}_1\mathbf{v}_1^T\mathbf{m}$$

$\mathbf{v}_1$ strong input direction $\mathbf{v}_2$ weak input direction
$\mathbf{u}_1$ strong output direction $\mathbf{u}_2$ weak input direction

Koung & MacGregor(1994) Design — $\boldsymbol{\xi}$ = rotated inputs

Based on latest $\hat{\mathbf{G}}$: $\mathbf{y}=\hat{\mathbf{U}}\hat{\mathbf{\Sigma}}\underbrace{\hat{\mathbf{V}}^T\mathbf{m}}_{\boldsymbol{\xi}}$

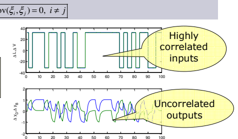
$\mathrm{var}(\xi_j)/\mathrm{var}(\xi_k)=\sigma_k^2/\sigma_j^2$

$\mathrm{cov}(\xi_i,\xi_j)=0,\ i\neq j$

Determine PRBS amplitudes for $\boldsymbol{\xi}$: $\mathbf{y}^{low}\le\mathbf{y}=\hat{\mathbf{u}}_1\hat{\sigma}_1\xi_1+\cdots+\hat{\mathbf{u}}_n\hat{\sigma}_n\xi_n\le\mathbf{y}^{high}$

Implement $\mathbf{m}=\hat{\mathbf{V}}\boldsymbol{\xi}$

$$\begin{bmatrix} m_1\\ m_2\end{bmatrix}=\mathbf{v}_1\xi_1+\mathbf{v}_2\xi_2\overset{\xi_2>>\xi_1}{\approx}\mathbf{v}_2\xi_2$$



Main Result
Integral Controllability Bound

$$\underbrace{\sum_{k=1}^{n}a_k\sqrt{\hat{\mathbf{v}}_k^T\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\hat{\mathbf{v}}_k}}_{\text{IC Bound}}<1\Rightarrow \mathrm{Re}\left[\lambda\left(\mathbf{G}\hat{\mathbf{G}}^{-1}\right)\right]>0$$

$$\sum_{k=1}^{n}a_k\sqrt{\hat{\mathbf{v}}_k^T\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\hat{\mathbf{v}}_k}<1$$

From ID: $\hat{\mathbf{G}}=\hat{\mathbf{U}}\hat{\Sigma}\hat{\mathbf{V}}^T\triangleq\sum_{k=1}^{n}\hat{\sigma}_k\hat{\mathbf{u}}_k\hat{\mathbf{v}}_k^T$, $a_k=c\dfrac{\|\hat{\mathbf{u}}_k\|_\infty}{\hat{\sigma}_k}$

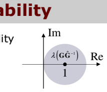
$$\mathbf{M}\triangleq\begin{bmatrix} m_{11} & \cdots & m_{n1}\\ \vdots & & \vdots\\ m_{1t} & \cdots & m_{nt}\end{bmatrix}$$

Design $\mathbf{M}$ or $\mathbf{M}^T\mathbf{M}$ to satisfy bound

Sidebar: Upper Bound that Ensures Integral Controllability

Only know $\mathbf{G}$ with uncertainty - Integral controllability satisfied if for all $\mathbf{D}$ in D

$$\sum_{k=1}^{n}\frac{\|\hat{\mathbf{u}}_k\|}{\hat{\sigma}_k}\|\mathbf{D}\hat{\mathbf{v}}_k\|<1$$



$$\mathbf{D}\triangleq\begin{bmatrix}\mathbf{d}_1^T\\ \vdots\\ \mathbf{d}_n^T\end{bmatrix}\triangleq\underbrace{\begin{bmatrix}\mathbf{g}_1^T\\ \vdots\\ \mathbf{g}_n^T\end{bmatrix}}_{\mathbf{G}}-\underbrace{\begin{bmatrix}\hat{\mathbf{g}}_1^T\\ \vdots\\ \hat{\mathbf{g}}_n^T\end{bmatrix}}_{\hat{\mathbf{G}}}$$

Use $D\triangleq\left\{\mathbf{D}\,|\,\mathbf{d}_k^T\mathbf{M}^T\mathbf{M}\mathbf{d}_k\le c^2, 1\le k\le n\right\}$

$c^2=s^2nF_{1-\alpha}(n,t-n)\approx\sigma_{noise}^2\chi_{1-\alpha}^2(n)$

Worst-case model uncertainty:

$$\max_{\mathbf{D}\in D}\left[\sum_{k=1}^{n}\frac{\|\hat{\mathbf{u}}_k\|_1}{\hat{\sigma}_k}\|\mathbf{D}\hat{\mathbf{v}}_k\|_\infty\right]=\sum_{k=1}^{n}\underbrace{c\frac{\|\hat{\mathbf{u}}_k\|_1}{\hat{\sigma}_k}}_{a_k}\sqrt{\hat{\mathbf{v}}_k^T\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\hat{\mathbf{v}}_k}$$

Select $\mathbf{M}$ or $\mathbf{M}^T\mathbf{M}$ to satisfy:

$$\sum_{k=1}^{n}a_k\sqrt{\hat{\mathbf{v}}_k^T\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\hat{\mathbf{v}}_k}<1$$

References:
Garcia, C.E. and Morari, M. (1985). Internal Model Control: 2. Design for Multivariable Systems. *Ind Eng. Chem. Process Des. Dev.*, 24, 472-484.
Koung, C.W. and MacGregor, J.F. (1994). Identification for Robust Multivariable Control: The Design of Experiments. *Automatica*, 30(10), 1541-1554.
Skogestad, S. and Morari, M. (1987). Configuration Selection for Distillation Control. AIChE Journal, 33, 1620-1635.
Darby, M.L. and Nikolaou, M. (Submitted for publication). Multivariable System Identification for Integral Controllability. *Automatica.*

Implications for Input Design

Let $\mathbf{M}^T\mathbf{M}=\mathbf{PAP}^T$, $\mathbf{\Lambda}=\begin{bmatrix}\lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2\end{bmatrix}$ Consider fixed Λ - fixed information content

Minimize worst case bound

$$\min_{\mathbf{P}\text{ orthonormal}}\left[\sum_{k=1}^{n}a_k\sqrt{\hat{\mathbf{v}}_k^T(\mathbf{M}^T\mathbf{M})^{-1}\hat{\mathbf{v}}_k}\right]=\min_{\mathbf{W}\text{ orthonormal}}\left[\sum_{k=1}^{n}a_k\sqrt{\mathbf{w}_k^T\mathbf{\Lambda}^{-1}\mathbf{w}_k}\right]=\sum_{k=1}^{n}\frac{a_k}{\lambda_{\ell_k}}\quad 0<a_1<\ldots<a_n,\ 0<\lambda_1<\ldots<\lambda_n$$

$\mathbf{P}^{opt}=\mathbf{\Pi}\hat{\mathbf{V}}$ (col $\hat{\mathbf{V}}$ rearranged)
Recover rotated inputs: $\boldsymbol{\xi}=\hat{\mathbf{V}}^T\mathbf{m}$
$\mathrm{cov}(\boldsymbol{\xi})=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\frac{1}{t-1}$ (uncorrelated ξ_k)

Integral controllability satisfied

$$\sum_{k=1}^{n}\frac{a_k}{\lambda_{\ell_k}}<1$$

Time to achieve integral controllability

$$\sum_{k=1}^{n}\frac{a_k}{\lambda_{\ell_k}}<1\Rightarrow T\ge\left(\sum_{k=1}^{n}a_k/\sqrt{\mathrm{var}(\xi_k)}\right)^2+1$$

Implications for Input Design: Selecting λ_k-Analytical Solutions

Motivated by MPC, lump input and output variances into a weighted sum

$$x\sum_{k=1}^{n}\mathrm{var}(m_i)+(1-x)\sum_{k=1}^{n}\mathrm{var}(y_i)\approx\frac{1}{t-1}\left(\sum_{k=1}^{n}\underbrace{x\lambda_{\ell_k}^2+(1-x)\sigma_k^2\lambda_{\ell_k}^2}_{b_k^2}\right),\ 0\le x\le 1$$

Consider following optimization problem:

Minimize variance s.t. IC bound

$$\min_{0<\lambda_{\ell_1}<\ldots<\lambda_{\ell_n}}\left\{\sum_{k=1}^{n}b_k^2\lambda_{\ell_k}^2\right\}\quad\text{s.t.}\ \sum_{k=1}^{n}\frac{a_k}{\lambda_{\ell_k}}\le 1-\varepsilon$$

Solution (satisfies: $0<\lambda_{\ell_1}<\ldots<\lambda_{\ell_n}$):

$$\lambda_k^{opt}=\frac{1}{1-\varepsilon}\left(\frac{a_k}{b_k^2}\right)^{1/3}\sum_{j=1}^{n}\left(a_jb_j\right)^{2/3}\Rightarrow\frac{\lambda_k^{opt}(x)}{\lambda_j^{opt}(x)}=\left(\frac{x\hat{\sigma}_j^3+(1-x)\hat{\sigma}_j}{x\hat{\sigma}_k^3+(1-x)\hat{\sigma}_k}\frac{\|\hat{\mathbf{u}}_k\|_1}{\|\hat{\mathbf{u}}_j\|_1}\right)^{1/3}\approx\left(\frac{x\hat{\sigma}_j^3+(1-x)\hat{\sigma}_j}{x\hat{\sigma}_k^3+(1-x)\hat{\sigma}_k}\right)^{1/3}$$

$x=1$ (output penalty only): $\dfrac{\lambda_j^2(1)}{\lambda_k^2(1)}\approx\dfrac{\sigma_k^2}{\sigma_j^2}$ — agrees with Koung-MacGregor

$x=0$ (input penalty only): $\dfrac{\lambda_j^2(1)}{\lambda_k^2(1)}\approx\left(\dfrac{\sigma_k^2}{\sigma_j^2}\right)^{1/3}$ — less aggressive than Koung-MacGregor

D-optimal design s.t. IC & variance bounds

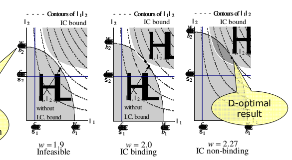
$$\max_{0<\lambda_{\ell_1}<\ldots<\lambda_{\ell_n}}\underbrace{\prod_{k=1}^{n}\lambda_{\ell_k}}_{\det\mathbf{M}^T\mathbf{M}}\quad\text{s.t.}\ \sum_{k=1}^{n}\frac{a_k}{\lambda_{\ell_k}}<\underbrace{1-\varepsilon}_{\delta},\ \sum_{k=1}^{n}b_k^2\lambda_{\ell_k}^2\le w^2$$

Solution (satisfies: $0<\lambda_{\ell_1}<\ldots<\lambda_{\ell_n}$)

IC bound inactive: $\lambda_{\ell_k}=\dfrac{w}{b_k\sqrt{n}}$ IC bound active: $r\left(\dfrac{a_k}{\lambda_{\ell_k}}-\dfrac{\delta}{n}\right)=b_k^2\lambda_{\ell_k}^2-\dfrac{w^2}{n}$, $\sum_{k=1}^{n}\dfrac{a_k}{\lambda_{\ell_k}}=\delta$, $r>0$

Example

2×2 case: $\sigma_1=1$, $\sigma_2=0.5$
$x=0.5$
(= input & output penalty)
$c=0.32$

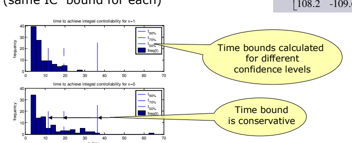


Use of Integral Controllability Bound to Determine Experimental Time

Using expression from earlier: $T=c^2\left(\sum_{k=1}^{n}\dfrac{\|\hat{\mathbf{u}}_k\|_1}{\hat{\sigma}_k\sqrt{\mathrm{var}(\xi_k)}}\right)^2+1$

Simulate 100 experiments based on min variance s.t. IC bound for x=0 & x=1 (same IC bound for each)

Skogestad/Morari Column $\mathbf{G}=\begin{bmatrix}87.8 & -86.4\\ 108.2 & -109.6\end{bmatrix}$



General Formulation for Input Design - Numerical Solution

Cannot assume $0<\lambda_{\ell_1}^{opt}<\ldots<\lambda_{\ell_n}^{opt}$ for general case: Solve numerically

- Split out time dependency
- Optimize wrt covariance of rotated inputs
- Time bound from optimal result

General Formulation

Non-Convex

$$\min_{\mathbf{w}_i,\bar{\lambda}_i}\sum_{k=1}^{n}c\frac{\|\hat{\mathbf{u}}_k\|_1}{\hat{\sigma}_k}\sqrt{\mathbf{w}_k^T\bar{\mathbf{\Lambda}}^{-1}\mathbf{w}_k}\qquad\text{Time bound: } T=c^2\mathrm{IC}_{opt}^2+1$$

st $\mathbf{w}_i^T\mathbf{w}_j=\delta_{ij}$ orthonormality
$[\hat{\mathbf{V}}\mathbf{W}^T\bar{\mathbf{\Lambda}}\mathbf{W}\hat{\mathbf{V}}^T]_{ii}\le\mathrm{var}(m_i)$ individual input variance bounds
$[\mathbf{T}\mathbf{W}^T\bar{\mathbf{\Lambda}}\mathbf{W}\mathbf{T}^T]_{ii}\le\mathrm{var}(y_i)$ individual output variance bounds

$\mathrm{cov}(\boldsymbol{\xi})\approx\mathbf{W}^T\bar{\mathbf{\Lambda}}\mathbf{W}$
$\mathrm{cov}(\mathbf{m})\approx\hat{\mathbf{V}}\mathbf{W}^T\bar{\mathbf{\Lambda}}\mathbf{W}\hat{\mathbf{V}}^T$
$\mathrm{cov}(\mathbf{y})\approx\mathbf{T}\mathbf{W}^T\bar{\mathbf{\Lambda}}\mathbf{W}\mathbf{T}^T$

$\mathbf{T}\triangleq[\mathbf{t}_1|\cdots|\mathbf{t}_n]=\underbrace{[\hat{\mathbf{u}}_1|\cdots|\hat{\mathbf{u}}_n]}_{\hat{\mathbf{U}}}\begin{bmatrix}\hat{\sigma}_1 & & \\ & \ddots & \\ & & \hat{\sigma}_n\end{bmatrix}$

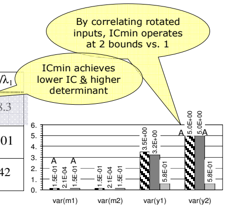
$\mathbf{W}\triangleq[\mathbf{w}_1|\cdots|\mathbf{w}_n]$

Example

$\mathbf{G}=\begin{bmatrix}87.8 & -86.4\\ 108.2 & -109.6\end{bmatrix}$ $\mathrm{var}(m_1)\le 0.15$, $\mathrm{var}(m_2)\le 1.5$, $\mathrm{var}(y_1)\le 5.0$, $\mathrm{var}(y_2)\le 5.0$

Design	IC	det(cov(**m**))	λ_2/λ_1
ICmin	1.30	6.08E-5	38.3
PRBS	4.70	4.44E-8	1.01
KM	2.05	4.47E-6	142

$$\mathrm{cov}(\boldsymbol{\xi}_{IC})\approx\bar{\Lambda}=\begin{bmatrix}2.04\text{E-}4 & 2.01\text{E-}4\\ 2.01\text{E-}4 & 2.99\text{E-}1\end{bmatrix}$$

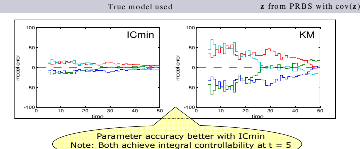


Time Realizations

$\mathbf{G}=\begin{bmatrix}87.8 & -86.4\\ 108.2 & -109.6\end{bmatrix}$

$\sigma_{noise}^2=1$ for both outputs
Simulate for 50 time steps
Start id at t = 5
True model used

Steady-state, no dynamics
Generate correlated rotated inputs: $\boldsymbol{\xi}=\mathbf{Q}^T\mathbf{z}$, $\mathbf{Q}^T\mathbf{Q}=\bar{\Lambda}$
$\mathbf{z}$ from PRBS with $\mathrm{cov}(\mathbf{z})=\mathbf{I}$



Ongoing Work

- Larger systems - FCC (5x5) example
- Dynamic case – apply PRBS design to **z** based on estimates of open-loop dynamics:

$$\mathrm{cov}(\mathbf{z})=\mathbf{I},\ \boldsymbol{\xi}=\mathbf{Q}^T\mathbf{z},\ \mathbf{Q}^T\mathbf{Q}=\bar{\Lambda}$$

- Constrained case with min/max outputs and inputs (not just variances)

Conclusions

- Mathematical framework for designing experiments for *nxn* systems
- No need to classify systems as ill- or well-conditioned
- Algebraic in nature, allowing rigorous formulation of optimization problems
- Recover previous design results and propose new ones
- Optimal experiments for typical situations lead to correlated rotated inputs

Future Work

- Application to adaptive design of experiments (do not know true plant)
- Rigorous extension to dynamic models
- Incorporate IC bound into closed-loop testing schemes
- Apply to non-square systems (meaningful for MPC)
- Distribution of experimental time, Probabilistic stability formulations